Relation between $\underline{S}$ and $\underline{Z}$

$\underline{Z}$: open-circuit imp. matrix of N-port

$\underline{Z}$ not diagonal

$$\underline{V} = \underline{Z}\underline{I}$$

$$\underline{R}^{\frac{1}{2}}(\underline{a}+\underline{b}) = \underline{Z}\underline{R}^{-\frac{1}{2}}(\underline{a}-\underline{b})$$

normalized

$$\underline{a}+\underline{b} = \underline{R}^{-\frac{1}{2}}\underline{Z}\underline{R}^{-\frac{1}{2}}(\underline{a}-\underline{b}) = \underline{Z}_n(\underline{a}-\underline{b}) \qquad \underline{Z}_n = \underline{R}^{-\frac{1}{2}}\underline{Z}\underline{R}^{-\frac{1}{2}}$$

$$z_{ij_n} = z_{ij} / \sqrt{R_i R_j}$$

$$(\underline{Z}_n + \underline{1})\underline{b} = (\underline{Z}_n - \underline{1})\underline{a}$$

$$\underline{b} = (\underline{Z}_n + \underline{1})^{-1}(\underline{Z}_n - \underline{1})\underline{a}$$

$$\boxed{\underline{S} = (\underline{Z}_n + \underline{1})^{-1}(\underline{Z}_n - \underline{1}) = (\underline{Z}_n - \underline{1})(\underline{Z}_n + \underline{1})^{-1}}$$

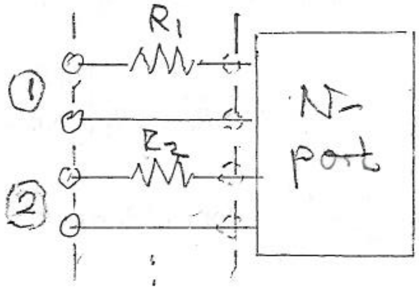
$$S = \frac{Z-R}{Z+R} = \frac{z_n - 1}{z_n + 1}$$

$$\underline{1} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix} = \underline{U}$$

$$\underline{S} = (\underline{Z}_n - \underline{1})(\underline{Z}_n + \underline{1})^{-1} = (\underline{Z}_n + \underline{1} - \underline{2})(\underline{Z}_n + \underline{1})^{-1}$$
$$= \underline{1} - 2(\underline{Z}_n + \underline{1})^{-1} = \underline{\underline{1 - 2Y_{an}}}$$

Where

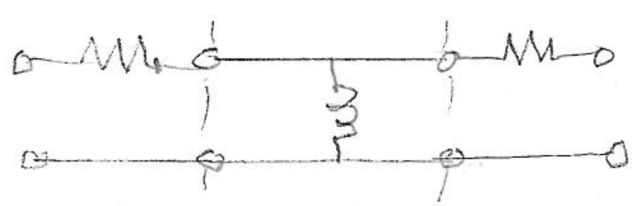
$$\underline{Y}_{an} = (\underline{Z}_n + \underline{1})^{-1} = (\underline{R}^{-\frac{1}{2}}\underline{Z}\underline{R}^{-\frac{1}{2}} + \underline{R}^{-\frac{1}{2}}\underline{R}\underline{R}^{-\frac{1}{2}})^{-1}$$
$$= \left[\underline{R}^{-\frac{1}{2}}(\underline{Z}+\underline{R})\underline{R}^{-\frac{1}{2}}\right]^{-1}$$
$$= \underline{R}^{\frac{1}{2}}(\underline{Z}+\underline{R})^{-1}\underline{R}^{\frac{1}{2}}$$



(Normalized & augmented $\underline{Y}$ matrix)

As $\underline{Y}_a$, and hence $\underline{Y}_{an}$, always exists for any well-defined n-port, it means that $\underline{S}$ also always exists.

$$\underline{Y}_n = \underline{R}^{1/2}\,\underline{Y}\,\underline{R}^{1/2} \qquad y_{n_{kl}} = y_{kl}\sqrt{R_l R_k}$$



$\underline{Y}$ doesn't exist, but $\underline{Y}_n$ does.

$$\underline{R}^{1/2} = \begin{bmatrix} R_1^{1/2} & & & 0 \\ & R_2^{1/2} & & \\ & & \ddots & \\ 0 & & & R_N^{1/2} \end{bmatrix}$$

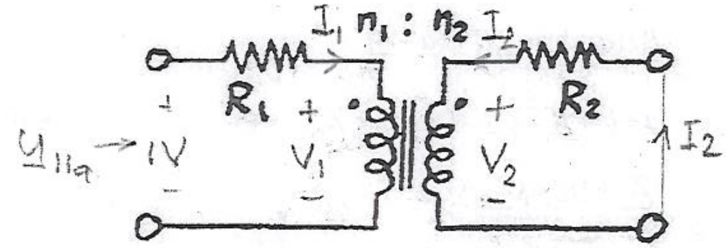
turns ratio

Examples

$n : 1$

Ideal transformer

$\frac{n_1}{n_2}$ turns ratio



$V_1 = \frac{n_1}{n_2} V_2$

$I_1 = -\frac{n_2}{n_1} I_2$

$\underset{\sim}{Y}, \underset{\sim}{Z}$ not exist

$$\underline{Y_a} = \frac{1}{n_2^2 R_1 + n_1^2 R_2}\begin{bmatrix} n_2^2 & -n_1 n_2 \\ -n_1 n_2 & n_1^2 \end{bmatrix}$$

$y_{11a} = \frac{1}{R_1 + \left(\frac{n_1}{n_2}\right)^2 R_2}$

$y_{ijn} = \sqrt{R_i R_j}\, y_{ij}$

$$\underline{Y_{an}} = \frac{1}{n_2^2 R_1 + n_1^2 R_2}\begin{bmatrix} n_2^2 R_1 & -n_1 n_2\sqrt{R_1 R_2} \\ -n_1 n_2 \sqrt{R_1 R_2} & n_1^2 R_2 \end{bmatrix}$$

$y_{21a} = \left.\frac{I_2}{V_1}\right|_{V_2=0} = -\frac{n_1}{n_2} I_1$

$$\underline{S} = \underline{1} - 2\underline{Y_{an}} = \frac{1}{n_2^2 R_1 + n_1^2 R_2}\begin{bmatrix} n_1^2 R_2 - n_2^2 R_1 & 2n_1 n_2\sqrt{R_1 R_2} \\ 2n_1 n_2 \sqrt{R_1 R_2} & n_2^2 R_1 - n_1^2 R_2 \end{bmatrix}$$

$I_1 = y_{11a}$

$I_2 = -\frac{n_1}{n_2} I_1 = \frac{-n_1/n_2}{R_1 + (\cdot)^2 R_2}$

If $\frac{R_1}{R_2} = \frac{n_1^2}{n_2^2}$ or $n_1^2 R_2 = n_2^2 R_1$

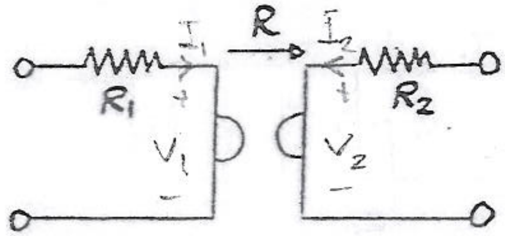
$y_{21a} = I_2 = y_{12a}$

Then $\underline{S} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

y_{22a} $1 \leftrightarrow 2$

Matched, lossless reciprocal circuit!

Gyrator



$$V_1 = z_{11} I_1 + z_{12} I_2$$
$$V_2 = z_{21} I_1 + z_{22} I_2$$

$$V_1 = -R I_2$$
$$V_2 = R I_1$$

$$\underset{\sim}{Z} = \begin{bmatrix} 0 & -R \\ R & 0 \end{bmatrix}$$

$$\underset{\sim}{Z_a} = \begin{bmatrix} R_1 & -R \\ R & R_2 \end{bmatrix}$$

$$\underset{\sim}{Z_a} = \underset{\sim}{Z} + \underset{\sim}{R}$$

$$\begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

$$\underline{Y_a} = \begin{bmatrix} R_1 & -R \\ R & R_2 \end{bmatrix}^{-1} = \frac{1}{R^2 + R_1 R_2} \begin{bmatrix} R_2 & R \\ -R & R_1 \end{bmatrix}$$

$$\underline{Y_{an}} = \frac{1}{R^2 + R_1 R_2} \begin{bmatrix} R_1 R_2 & R\sqrt{R_1 R_2} \\ -R\sqrt{R_1 R_2} & R_1 R_2 \end{bmatrix}$$

$$\underline{S} = \underline{1} - 2\underline{Y_{an}} = \frac{1}{R^2 + R_1 R_2} \begin{bmatrix} R^2 - R_1 R_2 & -2R\sqrt{R_1 R_2} \\ 2R\sqrt{R_1 R_2} & R^2 - R_1 R_2 \end{bmatrix}$$

If $R_1 R_2 = R^2$

nonreciprocal!

Then $\underline{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Note: We do not require $R_1 = R_2 = R$

Matched, lossless <u>non</u>reciprocal circuit

$$\underset{\sim}{1} = \frac{1}{R^2 + R_1 R_2} \begin{bmatrix} R^2 + R_1 R_2 & 0 \\ 0 & R^2 + R_1 R_2 \end{bmatrix}$$